

Rigidity of proper holomorphic mappings between certain unbounded non-hyperbolic domains

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Abstract. The Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ ($\mu > 0$) in \mathbf{C}^{n+m} is defined by the inequality $\|w\|^2 < e^{-\mu\|z\|^2}$, where $(z, w) \in \mathbf{C}^n \times \mathbf{C}^m$, which is an unbounded non-hyperbolic domain in \mathbf{C}^{n+m} . Recently, Yamamori gave an explicit formula for the Bergman kernel of the Fock-Bargmann-Hartogs domains in terms of the polylogarithm functions and Kim-Ninh-Yamamori determined the automorphism group of the domain $D_{n,m}(\mu)$. In this article, we obtain rigidity results on proper holomorphic mappings between two equidimensional Fock-Bargmann-Hartogs domains. Our rigidity result implies that any proper holomorphic self-mapping on the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ with $m \geq 2$ must be an automorphism.

Key words: Fock-Bargmann-Hartogs domains, Proper holomorphic mappings, Unbounded circular domains

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1 Introduction

In 1977, Alexander [2] proved the following fundamental result.

Theorem 1.A (Alexander [2]) *If $f : \mathbf{B}^n \rightarrow \mathbf{B}^n$ ($n \geq 2$) is a proper holomorphic self-mapping of the unit ball in \mathbf{C}^n , then f is an automorphism of \mathbf{B}^n .*

Alexander's theorem has been generalized to several classes of domains. Especially, there are many important results concerning proper holomorphic mapping $f : D_1 \rightarrow D_2$ between two bounded pseudoconvex domains D_1, D_2 in \mathbf{C}^n with smooth boundary. If the proper holomorphic mapping f extends smoothly to the closure of D_1 , then the extended mapping takes the boundary bD_1 into the boundary bD_2 , and it satisfies the tangential Cauchy-Riemann equations on bD_1 . Thus the proper holomorphic mapping $f : D_1 \rightarrow D_2$ leads naturally to the geometric study of the mappings from bD_1 into bD_2 . These researches are often heavily based on analytic techniques about the mapping on boundaries (e.g., see Forstnerič [9] and Huang [11] for references). In this regard, respectively, Diederich and Fornæss [8] and Bedford and Bell [3] proved the following results.

Theorem 1.B (Diederich and Fornæss [8]) *If $\Omega, D \subset \mathbf{C}^n$ ($n \geq 2$) are smoothly bounded pseudoconvex domains and Ω is strongly pseudoconvex, then any proper holomorphic mapping f of Ω into D is a local biholomorphism. Thus, if D is simply connected, then the mapping f is biholomorphic.*

Theorem 1.C (Bedford and Bell [3]) *Let D be bounded weakly pseudoconvex domain in \mathbf{C}^n ($n \geq 2$) with smooth real-analytic boundary. Then any proper holomorphic self-mapping of D is an automorphism.*

We remark that $f(z_1, z_2) = (z_1, z_2^2) : |z_1|^2 + |z_2|^4 < 1 \rightarrow |w_1|^2 + |w_2|^2 < 1$ is a proper holomorphic mapping between two bounded pseudoconvex domains in \mathbf{C}^2 with smooth real-analytic boundary, but it is branched and is not biholomorphic. Thus Theorem 1.C suggests a very interesting subject to discover some interesting bounded weakly pseudoconvex domains D_1, D_2 in \mathbf{C}^n ($n \geq 2$) such that any proper holomorphic mapping from D_1 to D_2 is a biholomorphism. Even though the bounded homogeneous domains in \mathbf{C}^n are always pseudoconvex, there are, of course, many such domains (e.g., all bounded symmetric domains of rank ≥ 2) without smooth boundary.

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The lack of boundary regularity usually presents a serious analytical difficulty. In 1984, by using the results of Bell [4] and Tumanov-Henkin [27], Henkin and Novikov [10] proved the following result (see Th.3.3 in Forstnerič [9] for references).

Theorem 1.D (Henkin and Novikov [10]) *Any proper holomorphic self-mapping on an irreducible bounded symmetric domain of rank ≥ 2 is an analytic automorphism.*

Further, using the idea in Mok-Tsai [19] and Tsai [24], Tu [25, 26] (one of the authors of the current article) and Mok-Ng-Tu [18] obtained some rigidity results of proper holomorphic mappings between equidimensional bounded symmetric domains (also called Cartan's domains). Recently, Ahn-Byun-Park [1] determined the automorphism group of the Cartan-Hartogs domains (also called extended Cartan's domains) over classical domains. In the past decade, Isaev [12], Isaev-Krantz [13] and Kim-Verdiani [16] also described the automorphism groups of hyperbolic domains.

The Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ are defined by

$$D_{n,m}(\mu) := \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\}, \quad \mu > 0.$$

The Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ are unbounded strongly pseudoconvex domains in \mathbf{C}^{n+m} . We note that each $D_{n,m}(\mu)$ contains $\{(z, 0) \in \mathbf{C}^n \times \mathbf{C}^m\} \cong \mathbf{C}^n$. Thus each $D_{n,m}(\mu)$ is not hyperbolic in the sense of Kobayashi and $D_{n,m}(\mu)$ can not be biholomorphic to any bounded domain in \mathbf{C}^{n+m} . Therefore, each Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ is an unbounded non-hyperbolic domain in \mathbf{C}^{n+m} .

In 2013, Yamamori [28] gave an explicit formula for the Bergman kernel of the Fock-Bargmann-Hartogs domains in terms of the polylogarithm functions. In 2014, by checking that the Bergman kernel ensures revised Cartan's theorem, Kim-Ninh-Yamamori [15] determined the automorphism group of the Fock-Bargmann-Hartogs domains as follows:

Theorem 1.E (Kim-Ninh-Yamamori [15]) *The automorphism group $\text{Aut}(D_{n,m}(\mu))$ is exactly the group generated by all automorphisms of $D_{n,m}(\mu)$ as follows:*

$$\varphi_U : (z, w) \mapsto (Uz, w), \quad U \in \mathcal{U}(n);$$

$$\varphi_{U'} : (z, w) \mapsto (z, U'w), \quad U' \in \mathcal{U}(m);$$

$$\varphi_v : (z, w) \mapsto (z + v, e^{-\mu\langle z, v \rangle - \frac{\mu}{2}\|v\|^2} w), \quad (v \in \mathbb{C}^n),$$

where $\mathcal{U}(k)$ is the unitary group of degree k , and $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on \mathbb{C}^n .

The purpose of this article is to prove the rigidity result on proper holomorphic mappings between equidimensional Fock-Bargmann-Hartogs domains as follows.

Theorem 1.1 *If $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ are two equidimensional Fock-Bargmann-Hartogs domains with $m \geq 2$ and f is a proper holomorphic mapping of $D_{n,m}(\mu)$ into $D_{n',m'}(\mu')$, then f is a biholomorphism between $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$.*

For example. Let $\Phi(z_1, \dots, z_n, w_1) := (\sqrt{2}z_1, \dots, \sqrt{2}z_n, w_1^2)$, $(z_1, \dots, z_n, w_1) \in D_{n,1}(\mu)$. Then Φ is a proper holomorphic self-mapping of $D_{n,1}(\mu)$, but it is branched and isn't an automorphism of $D_{n,1}(\mu)$. Then the assumption " $m \geq 2$ " in Theorem 1.1 cannot be removed. Also, this example implies that a proper holomorphic self-mapping of unbounded strongly pseudoconvex domain in \mathbf{C}^n ($n \geq 2$) is possibly not an automorphism, and therefore, in general, Theorem 1.C does not hold for unbounded strongly pseudoconvex domains in \mathbf{C}^n ($n \geq 2$).

Next we give a description of the biholomorphisms between two Fock-Bargmann-domains as follows:

Theorem 1.2 *Let $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ be two equidimensional Fock-Bargmann-Hartogs domains and let f be a biholomorphism between $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$. Then $n = n'$, $m = m'$ and therefore, there exists $\varphi \in \text{Aut}(D_{n',m'}(\mu'))$ such that*

$$f(z_1, \dots, z_n, w_1, \dots, w_m) = \varphi(\sqrt{\mu/\mu'}z_1, \dots, \sqrt{\mu/\mu'}z_n, w_1, \dots, w_m). \quad (1)$$

Now we shall present an outline of the argument in our proof of the main results. Let $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ be a proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains. In order to prove that $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ is a biholomorphism, it suffices to show that f is unbranched. Our proof consists of two steps:

The first is to prove that f extends holomorphically to their closures. The transformation rule for Bergman kernels under proper holomorphic mapping (e.g., Th. 1 in Bell [5]) is also valid for unbounded domain (e.g., see Cor. 1 in Trybula [23]). Note that the coordinate functions play a key role in the approach of Bell [5] to extend proper holomorphic mapping, but, in general, are no longer square integrable on unbounded domains. In order to overcome the difficulty, by combining the transformation rule for Bergman kernel under proper holomorphic mapping in Bell [5] and an explicit form of the Bergman kernel function for $D_{n,m}(\mu)$ in Yamamori [28], we use a kind of semi-regularity at the boundary of the Bergman kernel associated to $D_{n,m}(\mu)$ (see Th. 2.3 in this paper) to extend the proper map holomorphically to a neighborhood of the closure $\overline{D_{n,m}(\mu)}$ of $D_{n,m}(\mu)$, and then finish the first step.

The second is to prove that $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ is unbranched assuming that the first step is achieved. Assume that the zero locus S of the complex Jacobian of the proper holomorphic mapping f on $D_{n,m}(\mu)$ is not empty. Then S is of the codimension 1. To finish the second step, by using the strongly pseudoconvex boundary of $D_{n,m}(\mu)$ and the local regularity for the mappings between strongly pseudoconvex hypersurfaces (e.g., see Pinčuk [20]), we get $\overline{S} \cap bD_{n,m}(\mu) = \emptyset$ (note this will force S to be compact if $D_{n,m}(\mu)$ is bounded) and then S is a complex analytic subset of \mathbf{C}^{n+m} . Further, we get that the complex analytic subset S of \mathbf{C}^{n+m} must be an algebraic set by its growth estimates. And, by considering the dimension of the intersection of the projective closure \overline{S} of the affine algebraic set S with the hyperplane at infinity, we obtain that S is of the codimension $\geq m$, which forces S to be \emptyset by the assumption $m \geq 2$ in Theorem 1.1. Therefore, f is unbranched and is a biholomorphism. This is the key ideas in proving the main results.

Our main work implies that any proper holomorphic self-mapping on the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ with $m \geq 2$ must be an automorphism.

2 Preliminaries

2.1 Bergman kernel associated to $D_{n,m}(\mu)$

In this section we will make an investigation on a kind of semi-regularity at the boundary of the Bergman kernel associated to $D_{n,m}(\mu)$.

For a domain Ω in \mathbf{C}^n , let $A^2(\Omega)$ be the Hilbert space of square integrable holomorphic functions on Ω with the inner product:

$$\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} dV(z) \quad (f, g \in \mathcal{O}(\Omega)),$$

where dV is the Euclidean volume form. The Bergman kernel $K(z, w)$ of $A^2(\Omega)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega)$, that is, for all $f \in A^2(\Omega)$, we have

$$f(z) = \int_{\Omega} f(w) K(z, w) dV(w) \quad (z \in \Omega).$$

For a positive continuous function p on Ω , let $A^2(\Omega, p)$ be the weighted Hilbert space of square integrable holomorphic functions with respect to the weight function p with the inner product:

$$\langle f, g \rangle = \int_{\Omega} f(z) \overline{g(z)} p(z) dV(z) \quad (f, g \in \mathcal{O}(\Omega)).$$

Similarly, the weighted Bergman kernel $K_{A^2(\Omega, p)}$ of $A^2(\Omega, p)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega, p)$. For a positive integer m , define the Hartogs domain $\Omega_{m, p}$ over Ω by

$$\Omega_{m, p} = \{(z, w) \in \Omega \times \mathbf{C}^m : \|w\|^2 < p(z)\}.$$

Ligocka [17] showed that the Bergman kernel of $\Omega_{m, p}$ can be expressed as infinite sum in terms of the weighted Bergman kernel of $A^2(\Omega, p^k)$ ($k = 1, 2, \dots$) as follows.

Theorem 2.1 (Ligocka [17]) *Let K_m be the Bergman kernel of $\Omega_{m, p}$ and let $K_{A^2(\Omega, p^k)}$ be the weighted Bergman kernel of $A^2(\Omega, p^k)$ ($k = 1, 2, \dots$). Then*

$$K_m((z, w), (t, s)) = \frac{m!}{\pi^m} \sum_{k=0}^{\infty} \frac{(m+1)_k}{k!} K_{A^2(\Omega, p^{k+m})}(z, t) \langle w, s \rangle^k,$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1) \cdots (a+k-1)$.

The Fock-Bargmann space is the weighted Hilbert space $A^2(\mathbf{C}^n, e^{-\mu\|z\|^2})$ on \mathbf{C}^n with the Gaussian weight function $e^{-\mu\|z\|^2}$ ($\mu > 0$). The reproducing kernel of $A^2(\mathbf{C}^n, e^{-\mu\|z\|^2})$, called the Fock-Bargmann kernel, is $\mu^n e^{\mu\langle z, t \rangle} / \pi^n$ (see Bargmann [6]). In 2013, using Th. 2.1 and the expression of the Fock-Bargmann kernel, Yamamori [28] give the Bergman kernel of the Fock-Bargmann-Hartogs domain $D_{n, m}(\mu)$ as follows.

Theorem 2.2 (Yamamori [28]) *The Bergman kernel of the Fock-Bargmann-Hartogs domain $D_{n, m}(\mu)$ is given by*

$$\begin{aligned} & K_{D_{n, m}(\mu)}((z, w), (t, s)) \\ &= \frac{m! \mu^n}{\pi^{m+n}} \sum_{k=0}^{\infty} \frac{(m+1)_k (k+m)^n}{k!} e^{\mu(k+m)\langle z, t \rangle} \langle w, s \rangle^k \\ &= \frac{m! \mu^n}{\pi^{m+n}} \sum_{k=0}^{\infty} \frac{(m+1)_k (k+m)^n}{k!} e^{\mu(k+m)\langle z, t \rangle} \sum_{\alpha_1 + \dots + \alpha_m = k} \frac{k!}{\alpha_1! \cdots \alpha_m!} (w_1 \bar{s}_1)^{\alpha_1} \cdots (w_m \bar{s}_m)^{\alpha_m} \\ &= \frac{m! \mu^n}{\pi^{m+n}} \sum_{\alpha \in \mathbf{N}^m} \frac{(m+1)_{|\alpha|} (|\alpha| + m)^n}{\alpha!} e^{\mu(|\alpha| + m)\langle z, t \rangle} w^\alpha \bar{s}^\alpha, \end{aligned}$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1) \cdots (a+k-1)$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbf{N}^m$ are multi-indices of non-negative integer, $|\alpha| = \alpha_1 + \dots + \alpha_m$, $\alpha! = \alpha_1! \cdots \alpha_m!$ and $w^\alpha = w_1^{\alpha_1} \cdots w_m^{\alpha_m}$.

Now we give a kind of semi-regularity at the boundary of the Bergman kernel associated to $D_{n, m}(\mu)$ as follows:

Theorem 2.3 *Let $D_{n, m}(\mu)$ be a Fock-Bargmann-Hartogs domain and let $K_{D_{n, m}(\mu)}((z, w), (t, s))$ be its Bergman kernel. If E is a compact subset of $D_{n, m}(\mu)$, then there is an open set G containing $\overline{D_{n, m}(\mu)}$ such that for each $(t, s) \in E$, the function $K_{D_{n, m}(\mu)}((z, w), (t, s))$ extends to be holomorphic on G as a function of (z, w) .*

Proof. Since E is a compact subset of $D_{n, m}(\mu)$, there exists a real number r with $0 < r < 1$ such that $E \subset \{(z, w) \in D_{n, m}(\mu) : \|w\|^2 < r^2 e^{-\mu\|z\|^2}\}$. Let $G := \{(z, w) \in D_{n, m}(\mu) : \|w\|^2 < \frac{1}{r^2} e^{-\mu\|z\|^2}\}$. Then G is an open set containing $\overline{D_{n, m}(\mu)}$. By Theorem 2.2, we have

$$K_{D_{n, m}(\mu)}((z, w), (t, s)) = K_{D_{n, m}(\mu)}((z, rw), (t, \frac{1}{r}s))$$

for all $(z, w) \in D_{n, m}(\mu)$, $(t, s) \in \{(z, w) \in D_{n, m}(\mu) : \|w\|^2 < r^2 e^{-\mu\|z\|^2}\}$. Thus, for every fixed $(t, s) \in E$, $K_{D_{n, m}(\mu)}((z, w), (t, s))$ extends holomorphically to G as a function of (z, w) . The proof of Theorem 2.3 is finished.

2.2 Holomorphic extensions of proper holomorphic mappings

In this section we will use Bell's transformation rule for Bergman kernels under the proper holomorphic mappings and the semi-regularity at the boundary of the Bergman kernel associated to $D_{n,m}(\mu)$ to show that any proper holomorphic mapping f between two equidimensional Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ can be extended holomorphically to the closure $\overline{D_{n,m}(\mu)}$ of $D_{n,m}(\mu)$.

The transformation rule for Bergman kernels under the proper holomorphic mappings in Bell [5] plays an important role in holomorphic extensions of proper holomorphic mappings. The transformation rule (e.g., Th. 1 in Bell [5]) is also valid for unbounded domain (e.g., see Cor. 1 in Trybula [23]). Then we have the transformation rule for Bergman kernels under the proper holomorphic mappings as follows.

Theorem 2.4 (Bell [5], Theorem 1) *Suppose that Ω_1 and Ω_2 are two domains (not necessarily bounded) in \mathbf{C}^n and that f is a proper holomorphic mapping of Ω_1 onto Ω_2 of order r . Let $u = \det[f']$ and let F_1, F_2, \dots, F_r denote the r local inverses to f defined locally on $\Omega_2 \setminus S$ where $S = \{f(z) : u(z) = 0\}$. Let $U_k = \det[F'_k]$ and let $K_i(z, w)$ denote the Bergman kernel function associated to Ω_i for $i = 1, 2$. The Bergman kernels transform according to*

$$\sum_{k=1}^r K_1(z, F_k(w)) \overline{U_k(w)} = u(z) K_2(f(z), w) \quad (2)$$

for all $z \in \Omega_1$ and $w \in \Omega_2 \setminus S$.

Remark on Theorem 2.4. The removable singularity theorem states that if $V(\subsetneq D)$ is a complex variety in a domain D and $h \in L^2(D)$ (i.e., The Hilbert space of square integrable functions on D) is holomorphic on $D \setminus V$, then h is holomorphic on D . Then the function on the left-hand side of (2) extends to be antiholomorphic in w for all $w \in \Omega_2$ by the removable singularity theorem (see Bell [5] for references here).

Now we will use Bell's transformation rule for Bergman kernels and the semi-regularity at the boundary of the Bergman kernel associated to $D_{n,m}(\mu)$ to show the holomorphic extension theorem as follows.

Theorem 2.5 *If $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ are two equidimensional Fock-Bargmann-Hartogs domains and f is a proper holomorphic mapping of $D_{n,m}(\mu)$ into $D_{n',m'}(\mu')$, then f extends to be holomorphic in a neighborhood of $\overline{D_{n,m}(\mu)}$.*

Proof. Let f be a proper holomorphic mapping of $D_{n,m}(\mu)$ onto $D_{n',m'}(\mu')$ and $u = \det[f']$. A classical theorem due to R. Remmert (c.f. Rudin [21], Theorem 15.1.9) states that f is a branched covering of some finite order r and the set $S = \{f(z, w) \in D_{n',m'}(\mu') : u(z, w) = 0\}$ is a complex analytic variety in $D_{n',m'}(\mu')$.

Let F_1, F_2, \dots, F_r denote the r local inverses to f defined locally on $D_{n',m'}(\mu') \setminus S$. Let $U_k = \det[F'_k]$ and let $K_1((z, w), (t, s))$ and $K_2((z', w'), (t', s'))$ denote the Bergman kernel function associated to $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ respectively. Write

$$H((z, w), (t', s')) = \sum_{k=1}^r K_1((z, w), F_k(t', s')) \overline{U_k(t', s')}.$$

Then, from Remark on Theorem 2.4, we have that $H((z, w), (t', s'))$ is holomorphic in (z, w) and is antiholomorphic in (t', s') for all $((z, w), (t', s')) \in D_{n,m}(\mu) \times D_{n',m'}(\mu')$.

With this notation, the transformation formula (2) for Bergman kernels becomes

$$H((z, w), (t', s')) = u(z, w) K_2(f(z, w), (t', s')). \quad (3)$$

Write $f(z, w) = (f_1(z, w), f_2(z, w)) \in \mathbf{C}^{n'} \times \mathbf{C}^{m'}$. For $\alpha = (\alpha', \alpha'') \in \mathbf{N}^n \times \mathbf{N}^m$, write

$$H^{(\alpha)}((z, w), (t', s')) := \frac{\partial^\alpha}{\partial t'^{\alpha'} \partial s'^{\alpha''}} H((z, w), (t', s')).$$

By differentiating the equation (3) with respect to $(\overline{t'}, \overline{s'})$, from Theorem 2.2, we have

$$\begin{aligned}
H^{(\alpha)}((z, w), (t', s')) &= u(z, w) \frac{\partial^\alpha}{\partial \overline{t'}^{\alpha'} \partial \overline{s'}^{\alpha''}} K_2(f(z, w), (t', s')) \\
&= u(z, w) \frac{\partial^\alpha}{\partial \overline{t'}^{\alpha'} \partial \overline{s'}^{\alpha''}} \left[\frac{m'! \mu'^{n'}}{\pi^{m'+n'}} \sum_{\beta \in \mathbf{N}^{m'}} \frac{(m'+1)_{|\beta|} (|\beta| + m')^{n'}}{\beta!} e^{\mu'(|\beta| + m') \langle f_1(z, w), t' \rangle} f_2(z, w)^{\beta} \overline{s'}^\beta \right] \\
&= u(z, w) \frac{m'! \mu'^{n'}}{\pi^{m'+n'}} \sum_{\beta: \beta - \alpha'' \in \mathbf{N}^{m'}} \frac{(m'+1)_{|\beta|} (|\beta| + m')^{n'}}{\beta!} \mu'^{|\alpha'|} (|\beta| + m')^{|\alpha'|} f_1(z, w)^{\alpha'} e^{\mu'(|\beta| + m') \langle f_1(z, w), t' \rangle} \\
&\quad \times \frac{\beta!}{(\beta - \alpha'')!} f_2(z, w)^{\beta} \overline{s'}^{\beta - \alpha''}.
\end{aligned}$$

By putting $(t', s') = (0, 0)$ in the above formula, we get

$$\begin{aligned}
H^{(\alpha)}((z, w), (0, 0)) &= u(z, w) \frac{m'! \mu'^{n'}}{\pi^{m'+n'}} \frac{(m'+1)_{|\alpha''|} (|\alpha''| + m')^{n'}}{\alpha''!} \mu'^{|\alpha'|} (|\alpha''| + m')^{|\alpha'|} f_1(z, w)^{\alpha'} \alpha''! f_2(z, w)^{\alpha''} \\
&= u(z, w) \frac{m'! \mu'^{n'+|\alpha'|}}{\pi^{m'+n'}} (m'+1)_{|\alpha''|} (|\alpha''| + m')^{n'+|\alpha'|} f(z, w)^\alpha \\
&= C(\alpha) u(z, w) f(z, w)^\alpha
\end{aligned}$$

for all $\alpha = (\alpha', \alpha'') \in \mathbf{N}^{n'} \times \mathbf{N}^{m'}$.

Fix a neighborhood V of $(0, 0)$ with $V \subset \subset D_{n', m'}(\mu')$. Then, for each $(t', s') \in \overline{V} \setminus S$, we have $F_k(t', s') \in f^{-1}(\overline{V}) \subset \subset D_{n, m}(\mu)$ ($1 \leq k \leq r$). Therefore, for all $(t', s') \in \overline{V} \setminus S$, by Theorem 2.3, we have $H((z, w), (t', s')) = \sum_{k=1}^r K_1((z, w), F_k(t', s')) \overline{U_k(t', s')}$ can extend holomorphically to a neighborhood G of the closure $\overline{D_{n, m}(\mu)}$ of $D_{n, m}(\mu)$ as a function of (z, w) . Hence $H((z, w), (t', s'))$ is holomorphic in (z, w) and anti-holomorphic in (t', s') for $((z, w), (t', s')) \in G \times (V \setminus S)$.

Therefore, we have that $H((z, w), (t', s'))$ is holomorphic in (z, w) and anti-holomorphic in (t', s') for all $((z, w), (t', s')) \in D_{n, m}(\mu) \times V$ (note $V \subset D_{n', m'}(\mu')$) and for all $((z, w), (t', s')) \in G \times (V \setminus S)$. So the Hartogs-type extension theorem implies that $H((z, w), (t', s'))$ can be extended to be a function on $G \times V$ which is holomorphic in (z, w) and anti-holomorphic in (t', s') for all $((z, w), (t', s')) \in G \times V$.

Hence $H^{(\alpha)}((z, w), (0, 0))$ can extend holomorphically to the neighborhood G of $\overline{D_{n, m}(\mu)}$ of $D_{n, m}(\mu)$ as a function of (z, w) for all $\alpha \in \mathbf{N}^{n+m}$. Thus, the function $u \cdot f^\alpha$ always extends holomorphically to the neighborhood G of $\overline{D_{n, m}(\mu)}$ for each $\alpha \in \mathbf{N}^{n+m}$. This implies that f extends to be holomorphic in the neighborhood G because the ring of germs of holomorphic functions is a unique factorization domain. The proof of Lemma 2.5 is finished.

2.3 Cartan's theorem revisited

Let $D \subset \mathbf{C}^N$ be a domain (not necessarily bounded) with $0 \in D$. Let $K_D(z, w)$ ($z, w \in D$) be the Bergman kernel of D . Let $T_D(z, w)$ be an $N \times N$ matrix defined by

$$T_D(z, w) := \begin{pmatrix} \frac{\partial^2}{\partial \overline{w}_1 \partial z_1} \log K_D(z, w) & \cdots & \frac{\partial^2}{\partial \overline{w}_1 \partial z_N} \log K_D(z, w) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2}{\partial \overline{w}_N \partial z_1} \log K_D(z, w) & \cdots & \frac{\partial^2}{\partial \overline{w}_N \partial z_N} \log K_D(z, w) \end{pmatrix}.$$

It is obviously that $K_D(0, 0) > 0$ and $T_D(0, 0)$ is positive definite for any bounded domain $D \subset \mathbf{C}^N$.

Ishi-Kai [14] proved Cartan's theorem by using the notion of the Bergman representative mapping for bounded circular domains. However their proof is obviously applicable for an unbounded domain whenever its Bergman kernel has some properties. Following the idea, Kim-Ninh-Yamamori [15] obtained a version of Cartan's theorem for an unbounded circular domain $D \subset \mathbf{C}^N$ such that

$K_D(0,0) > 0$ and $T_D(0,0)$ is positive definite, which assures that any automorphism f of such an unbounded circular domain with $f(0) = 0$ must be linear. In this section we will get a slight generalization of the result which states that any biholomorphism f between such two unbounded circular domains with $f(0) = 0$ must be linear.

Lemma 2.6 (Ishi-Kai [14], Prop. 2.1) *Let D_k be a circular domain (not necessarily bounded) in \mathbf{C}^N with $0 \in D_k$ ($k = 1, 2$). Let $\varphi : D_1 \rightarrow D_2$ be a biholomorphism with $\varphi(0) = 0$. If $K_{D_k}(0,0) > 0$ and $T_{D_k}(0,0)$ is positive definite ($k = 1, 2$), then φ is linear.*

Remark. see Theorem 4 in Kim-Ninh-Yamamori [15] for references here.

By the Lemmas 5 and 6 in Kim-Ninh-Yamamori [15], we have that each Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ satisfies the conditions that $K_{D_{n,m}(\mu)}(0,0) > 0$ and $T_{D_{n,m}(\mu)}(0,0)$ is positive definite. Therefore, by Lemma 2.6, we have a generalized Cartan's theorem for Fock-Bargmann-Hartogs domains as follows:

Theorem 2.7 *Let $\varphi : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ be a biholomorphism between two equidimensional Fock-Bargmann-Hartogs domains with $\varphi(0) = 0$. Then φ is linear.*

2.4 Some lemmas about complex analytic sets

In order to study the zero locus of the complex Jacobian of the proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains, we need the following results.

Lemma 2.8 (Chirka [7], §7.4 Theorem 3) *A pure p -dimensional analytic subset $A \subset \mathbf{C}^n$ is algebraic if and only if it is contained, after some unitary change of coordinates, in a domain $D : \|z''\| < C(1 + \|z'\|)^s$, where $z = (z', z'')$, $z' = (z_1, \dots, z_p)$, and C, s are certain constants.*

Lemma 2.9 (Chirka [7], §7.2 Proposition 2) *The closure in \mathbf{P}^n of an affine algebraic set $A = \{\zeta \in \mathbf{C}^n : p(\zeta) = 0\}$, where p is a polynomial of degree s , coincides with the projective algebraic set $\{[z] \in \mathbf{P}^n : p^*(z) = 0\}$, where p^* is the projectivization of p .*

In order to estimate the dimension of the zero locus of the complex Jacobian of the proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains, we need the following formula for the dimension of the intersection of two algebraic sets.

Lemma 2.10 (see Shafarevich [22]) *Let $X, Y \subset \mathbf{P}^N$ be irreducible quasiprojective varieties with $\dim X = n$ and $\dim Y = m$. Then any (nonempty) component Z of $X \cap Y$ has $\dim Z \geq n + m - N$.*

In order to prove our main conclusion, we need the preliminary lemma about regularity for the mappings between strongly pseudoconvex hypersurfaces due to Pinčuk [20] as follows.

Lemma 2.11 (Pinčuk [20], Lemma 1.3) *Let $D_1, D_2 \subset \mathbf{C}^n$ be two domains, $p \in bD_1$, and let U be a neighborhood of p in \mathbf{C}^n such that $U \cap \overline{D_1}$ is connected. Suppose that the mapping $f = (f_1, \dots, f_n) : U \cap \overline{D_1} \rightarrow \mathbf{C}^n$ is continuously differentiable in $U \cap \overline{D_1}$ and holomorphic in $U \cap D_1$ with $f(U \cap bD_1) \subset bD_2$. Take a domain $V \subset \mathbf{C}^n$ with $f(U \cap D_1) \subset V$. Suppose that $U \cap bD_1$ and $U \cap bD_2$ are strongly pseudoconvex hypersurfaces in \mathbf{C}^n . Then either f is constant or the Jacobian $J_f(z) = \det(\frac{\partial f_i}{\partial z_j})$ does not vanish in $U \cap bD_1$.*

3 Proof of main results

Proof of Theorem 1.1.

Let $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ be a proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ with $m \geq 2$. By Theorem 2.5, f extends holomorphically to a neighborhood V of $\overline{D_{n,m}(\mu)}$ with

$$f(bD_{n,m}(\mu)) \subset bD_{n',m'}(\mu'). \quad (4)$$

Define

$$A := \{\zeta \in V : J_f(\zeta) = 0\},$$

where $J_f(\zeta) = \det(\partial f_i / \partial \zeta_j)(\zeta)$ is the complex Jacobian determinant of

$$f(\zeta) := (f_1(\zeta), \dots, f_{n'+m'}(\zeta)) \quad (\zeta \in V).$$

Since $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$ are strongly pseudoconvex domains, the Jacobian $J_f(\zeta)$ does not vanish on $bD_{n,m}(\mu)$ by Lemma 2.11. Then we have $A \cap bD_{n,m}(\mu) = \emptyset$. Let $S := A \cap D_{n,m}(\mu)$. Therefore, we have

$$S \subset D_{n,m}(\mu), \quad \overline{S} \cap bD_{n,m}(\mu) = \emptyset. \quad (5)$$

If $S \neq \emptyset$, by (5), we can view S as a complex analytic set defined in \mathbf{C}^{n+m} . Moreover, for each $p \in S \subset D_{n,m}(\mu)$, we have

$$|w_m(p)|^2 \leq \|w(p)\|^2 < e^{-\mu\|z(p)\|^2} \leq 1 \leq 1 + \|(z, w')(p)\|,$$

where $w = (w', w_m)$. That is, we have

$$S \subset \{(z, w', w_m) \in \mathbf{C}^{n+m} : |w_m| < (1 + \|(z, w')\|)^{1/2}\}.$$

Therefore, by Lemma 2.8, we have that S must be an algebraic set of \mathbf{C}^{n+m} . Take an irreducible component S' of S . Now we consider the closure $\overline{S'}$ of S' in \mathbf{P}^{n+m} . By Lemma 2.9, $\overline{S'}$ is an projective algebraic set and $\dim \overline{S'} = \dim S' = \dim S = n + m - 1$.

Now we use Lemma 2.10 to give an upper bound n for $\dim S'$ and get a contradiction with $\dim S' = n + m - 1$. Let $[\zeta, z_1, \dots, z_n, w_1, \dots, w_m]$ be the homogeneous coordinate in \mathbf{P}^{n+m} and embed \mathbf{C}^{n+m} into \mathbf{P}^{n+m} as the affine piece $U_0 = \{[\zeta, z, w] \in \mathbf{P}^{n+m}, \zeta \neq 0\}$ by

$$(z_1, \dots, z_n, w_1, \dots, w_m) \hookrightarrow [1, z_1, \dots, z_n, w_1, \dots, w_m].$$

Then

$$D_{n,m}(\mu) \cap U_0 = \left\{ [\zeta, z, w] \in \mathbf{P}^{n+m} : \zeta \neq 0, \frac{\|w\|^2}{|\zeta|^2} < e^{-\mu \frac{\|z\|^2}{|\zeta|^2}} \right\}. \quad (6)$$

Let $H = \mathbf{P}^{n+m} \setminus \mathbf{C}^{n+m}$ be the hyperplane at infinity, that is $H = \{\zeta = 0\} \subset \mathbf{P}^{n+m}$. Consider the affine piece $U_1 = \{[\zeta, z, w] \in \mathbf{P}^{n+m}, z_1 \neq 0\}$ of \mathbf{P}^{n+m} with affine coordinate $(\xi, \lambda_2, \dots, \lambda_n, \eta_1, \dots, \eta_m)$. Then $\xi = \frac{\zeta}{z_1}, \lambda_2 = \frac{z_2}{z_1}, \dots, \lambda_n = \frac{z_n}{z_1}, \eta_1 = \frac{w_1}{z_1}, \dots, \eta_m = \frac{w_m}{z_1}$. Since $\frac{\|w\|^2}{|\zeta|^2} = \frac{\|w\|^2}{|z_1|^2} \left| \frac{z_1}{\zeta} \right|^2 = \frac{\|\eta\|^2}{|\xi|^2}$ and $e^{-\mu \frac{\|z\|^2}{|\zeta|^2}} = e^{-\mu \frac{\|z\|^2}{|z_1|^2} \left| \frac{z_1}{\zeta} \right|^2} = e^{-\mu \frac{(1+|\lambda_2|^2+\dots+|\lambda_n|^2)}{|\xi|^2}}$, by (6), we have

$$\begin{aligned} & D_{n,m}(\mu) \cap U_1 \cap U_0 \\ &= \left\{ (\xi, \lambda_2, \dots, \lambda_n, \eta_1, \dots, \eta_m) \in \mathbf{C}^{n+m} : |\eta_1|^2 + \dots + |\eta_m|^2 < |\xi|^2 e^{-\mu \frac{1+|\lambda_1|^2+\dots+|\lambda_n|^2}{|\xi|^2}} \right\}. \end{aligned} \quad (7)$$

Let $S'_1 = \overline{S'} \cap U_1$ be the affine piece of $\overline{S'}$ in U_1 and let $H_1 = H \cap U_1 = \{\xi = 0\}$ be the affine piece of the projective hyperplane H in U_1 . For each $p \in S'_1 \cap H_1$, there exists a sequence of points $\{p_k\} \subset \overline{S'} \cap ((U_1 \cap U_0) \setminus H_1)$ such that $p_k \rightarrow p$ ($k \rightarrow \infty$). Since $\{p_k\} \subset D_{n,m}(\mu) \cap U_1 \cap U_0$, by (7), we have

$$\|\eta(p_k)\|^2 < |\xi(p_k)|^2 e^{-\mu \frac{1+|\lambda_2(p_k)|^2+\dots+|\lambda_n(p_k)|^2}{|\xi(p_k)|^2}}. \quad (8)$$

Since $p \in H$, we have $\xi(p) = 0$ and $\xi(p_k) \rightarrow 0$ ($k \rightarrow \infty$). Let $k \rightarrow \infty$ in (8), we get $\|\eta(p)\|^2 = 0$. Therefore, $S'_1 \cap H_1 \subset \{\xi = 0, \eta_1 = \dots = \eta_m = 0\}$. Hence, $\dim(S'_1 \cap H_1) \leq n - 1$.

Further, by Lemma 2.10, we have

$$n - 1 \geq \dim(S'_1 \cap H_1) \geq \dim S'_1 + \dim H_1 - (n + m) = \dim S'_1 - 1.$$

Thus, $\dim S'_1 \leq n$.

Therefore, $n + m - 1 = \dim S = \dim S' = \dim \overline{S'} = \dim S'_1 \leq n$. Hence, $m \leq 1$, this is a contradiction with the assumption $m \geq 2$ of Theorem 1.1. This means $S = \emptyset$.

Thus $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ is unbranched. Since each Fock-Bargmann-Hartogs domain is simply connected, we get that $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ is a biholomorphism. The proof of Theorem 1.1 is finished.

Proof of Theorem 1.2.

For the completeness, here we will not assume Theorem 1.E to prove Theorem 1.2. Our proof of Theorem 1.2 is divided as two steps:

Step 1. Let $f : D_{n,m}(\mu) \rightarrow D_{n',m'}(\mu')$ be a biholomorphical mapping between two equidimensional Fock-Bargmann-Hartogs domains $D_{n,m}(\mu)$ and $D_{n',m'}(\mu')$. We will show that $n = n', m = m'$ and there exists a $\varphi' \in \text{Aut}(D_{n',m'}(\mu'))$ such that $\varphi' \circ f$ is a linear isomorphism.

Let $\mathcal{V} = \{(z, 0) \in \mathbf{C}^n \times \mathbf{C}^m\} \subset D_{n,m}(\mu)$ and $\mathcal{V}' = \{(z', 0) \in \mathbf{C}^{n'} \times \mathbf{C}^{m'}\} \subset D_{n',m'}(\mu')$. Put $f(z, 0) = (g(z), h(z))$ and $h(z) = (h_1(z), \dots, h_{m'}(z))$. Then we have

$$\sum_{i=1}^{m'} |h_i(z)|^2 = \|h(z)\|^2 < e^{-\mu' \|g(z)\|^2} \leq 1. \quad (9)$$

It follows that h_i is a bounded holomorphic function on \mathbf{C}^n for all $1 \leq i \leq m'$. Then Liouville's theorem implies that h_i is constant. Since g is a non-constant entire function, g is unbounded. Therefore, by (9), h must be identically equal to zero. This means $f(\mathcal{V}) \subset \mathcal{V}'$. In a similar way we have $f^{-1}(\mathcal{V}') \subset \mathcal{V}$. Thus, $f|_{\mathcal{V}}$ is a biholomorphism between \mathcal{V} and \mathcal{V}' . Therefore, we have $n = n', m = m'$.

Let $f(0, 0) = (v, \tilde{v})$. Then $\tilde{v} = 0$. Let $\varphi_{-v} \in \text{Aut}(D_{n',m'}(\mu'))$ be defined by

$$\varphi_{-v}(z', w') = (z' - v, e^{-\mu' \langle z', -v \rangle - \frac{\mu'}{2} \| -v \|^2} w').$$

Then $\varphi_{-v} \circ f(0, 0) = (0, 0)$. By Theorem 2.7, $\varphi_{-v} \circ f$ is a linear mapping.

Step 2. We now prove that there exist $\varphi^* \in \text{Aut}(D_{n',m'}(\mu'))$ such that $\varphi^* \circ \varphi_{-v} \circ f$ has the desired form (1).

Since $\varphi_{-v} \circ f$ is linear, the map $\varphi_{-v} \circ f$ can be written as a matrix form, namely,

$$\varphi_{-v} \circ f(z, w) = \begin{pmatrix} A & C \\ D & B \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix},$$

where $A \in M_{n \times n}(\mathbf{C}), B \in M_{m \times m}(\mathbf{C}), C \in M_{n \times m}(\mathbf{C})$ and $D \in M_{m \times n}(\mathbf{C})$. Since $\varphi_{-v} \circ f(\mathcal{V}) \subset \mathcal{V}'$, we have

$$D = \mathbf{0} \in M_{m \times n}(\mathbf{C}); \quad \det A \neq 0; \quad \det B \neq 0.$$

Then it follows that $\varphi_{-v} \circ f(z, w) = (Az + Cw, Bw)$ for all $(z, w) \in D_{n,m}(\mu)$, and $(\varphi_{-v} \circ f)^{-1}(z', w') = (A^{-1}z' - A^{-1}CB^{-1}w', B^{-1}w')$ for all $(z', w') \in D_{n',m'}(\mu')$.

Now we prove that $B \in \mathcal{U}(m)$. For $\|w\| < 1$, we have $(0, w) \in D_{n,m}(\mu)$ and $\varphi_{-v} \circ f(0, w) = (Cw, Bw) \in D_{n',m'}(\mu')$. Thus $\|Bw\| < 1$. On the other hand, for $\|w'\| < 1$, we have $(0, w') \in D_{n',m'}(\mu')$ and $(\varphi_{-v} \circ f)^{-1}(0, w') = (-A^{-1}CB^{-1}w', B^{-1}w') \in D_{n,m}(\mu)$. Thus $\|B^{-1}w'\| < 1$. Therefore,

$$B : \mathbf{B}^m \rightarrow \mathbf{B}^m, \quad w \rightarrow Bw$$

is a linear automorphism of the unit ball \mathbf{B}^m in \mathbf{C}^m , and thus $B \in \mathcal{U}(m)$.

Next we prove that $C = \mathbf{0} \in M_{n \times m}(\mathbf{C})$. For $\|w\| = 1$, we have $(0, w) \in bD_{n,m}(\mu)$. Since $\varphi_{-v} \circ f$ is a linear biholomorphism, we have $\varphi_{-v} \circ f(bD_{n,m}(\mu)) = bD_{n',m'}(\mu')$. Hence, $\varphi_{-v} \circ f(0, w) = (Cw, Bw) \in bD_{n',m'}(\mu')$. Therefore, by $B \in \mathcal{U}(m)$, we have $1 = \|w\|^2 = \|Bw\|^2 = e^{-\mu' \|Cw\|^2}$. Thus, $Cw = 0$ for all $\|w\| = 1$. Hence, $C = \mathbf{0} \in M_{n \times m}(\mathbf{C})$.

To complete our proof, it suffices to show that $A = \sqrt{\mu/\mu'}U$ for some $U \in \mathcal{U}(n)$. For any $z \in \mathbf{C}^n$, we can take $w \in \mathbf{C}^m$ such that $(z, w) \in bD_{n,m}(\mu)$. Since $\varphi_{-v} \circ f(bD_{n,m}(\mu)) = bD_{n',m'}(\mu')$ and $C = \mathbf{0}$, we have $\varphi_{-v} \circ f(z, w) = (Az, Bw) \in bD_{n',m'}(\mu')$. Thus, by $B \in \mathcal{U}(m)$, we get $e^{-\mu' \|Az\|^2} = \|Bw\|^2 = \|w\|^2 = e^{-\mu \|z\|^2}$. Therefore, $\|\sqrt{\mu'/\mu}Az\| = \|z\|$ for all $z \in \mathbf{C}^n$. Hence, $U = \sqrt{\mu'/\mu}A \in \mathcal{U}(n)$ and $A = \sqrt{\mu/\mu'}U$.

Let $\varphi_{U^{-1}}, \varphi_{B^{-1}} \in \text{Aut}(D_{n',m'}(\mu'))$ be defined by

$$\varphi_{U^{-1}}(z', w') = (U^{-1}z', w'), \quad \varphi_{B^{-1}}(z', w') = (z', B^{-1}w').$$

Then $\varphi := \varphi_{U^{-1}} \circ \varphi_{B^{-1}} \circ \varphi_{-v} \in \text{Aut}(D_{n',m'}(\mu'))$, and we have

$$\varphi \circ f(z_1, \dots, z_n, w_1, \dots, w_m) = (\sqrt{\mu/\mu'}z_1, \dots, \sqrt{\mu/\mu'}z_n, w_1, \dots, w_m).$$

The proof of Theorem 1.2 is finished.

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